Module 5

Mathematical Tools: Determinants and Matrices

Learning Objectives

After completing this module, students will be able to:

1. Understand how matrices and determinants are used as mathematical tools in QA.
2. Compute the value of a determinant.
3. Solve simultaneous equations with determinants.
4. Add, subtract, multiply, and divide matrices.
5. Transpose and find the inverse of matrices.
6. Use matrices to represent a system of equations.

Module Outline

M5.1 INTRODUCTION

Two new mathematical concepts, matrices and determinants, are introduced in this module. These tools are especially useful in Chapter 16 and the Supplement to Chapter 1, which deal with Markov analysis and game theory, but they are also handy computational aids for many other quantitative analysis problems, including linear programming, the topic of Chapters 7, 8, and 9.

M5.2 MATRICES AND MATRIX OPERATIONS

A *matrix* is an array of numbers arranged in rows and columns. Matrices, which are usually enclosed in parentheses or brackets, are used as an effective means of presenting or summarizing business data.

The following 2-row by 3-column (2×3) matrix, for example, might be used by television station executives to describe the channel switching behavior of their five o'clock TV news audience.

CURRENT STATION	AUDIENCE SWITCHING PROBABILITIES, NEXT MONTH'S ACTIVITY: CHANNEL 6	CHANNEL 8	STOP VIEWING
Channel 6	0.80	0.15	0.05
Channel 8	0.20	0.70	0.10

2 × 3 matrix

The number in the first row and first column indicates that there is a 0.80 probability that someone currently watching the Channel 6 news will continue to do so next month. Similarly, 15% of Channel 6's viewers are expected to switch to Channel 8 next month (row 1, column 2), 5% will not be watching the 5 o'clock news at all (row 1, column 3), and so on for the second row.

The remainder of this module deals with the numerous mathematical operations that can be performed on matrices. These include matrix addition, subtraction, and multiplication; transposing a matrix; finding its determinant, cofactors, and adjoint; and matrix inversion.

Matrix Addition and Subtraction

Matrix addition and *subtraction* are the easiest operations. Matrices of the same dimensions, that is, the same number of rows and columns, can be added or subtracted by adding or subtracting the numbers in the same row and column of each matrix. Here are two small matrices:

Adding and subtracting numbers

$$\text{matrix } A = \begin{pmatrix} 5 & 7 \\ 2 & 1 \end{pmatrix}$$

$$\text{matrix } B = \begin{pmatrix} 3 & 6 \\ 3 & 8 \end{pmatrix}$$

To find the sum of these 2×2 matrices, we add corresponding elements to create a new matrix.

$$\text{matrix } C = \text{matrix } A + \text{matrix } B$$

$$= \begin{pmatrix} 5 & 7 \\ 2 & \textcircled{1} \end{pmatrix} + \begin{pmatrix} 3 & 6 \\ 3 & \textcircled{8} \end{pmatrix} = \begin{pmatrix} 8 & 13 \\ 5 & \textcircled{9} \end{pmatrix}$$

To subtract matrix B from matrix A, we simply subtract the corresponding elements in each position.

$$\text{matrix } C = \text{matrix } A - \text{matrix } B$$

$$= \begin{pmatrix} \textcircled{5} & 7 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} \textcircled{3} & 6 \\ 3 & 8 \end{pmatrix} = \begin{pmatrix} \textcircled{2} & 1 \\ -1 & -7 \end{pmatrix}$$

Matrix Multiplication

Matrix multiplication is an operation that can take place only if the number of columns in the first matrix equals the number of rows in the second matrix. Thus, matrices of the dimensions in the following table can be multiplied:

MATRIX *A* SIZE	MATRIX *B* SIZE	SIZE OF *A* × *B* RESULTING
3 × 3	3 × 3	3 × 3
3 × 1	1 × 3	3 × 3
3 × 1	1 × 1	3 × 1
2 × 4	4 × 3	2 × 3
6 × 9	9 × 2	6 × 2
8 × 3	3 × 6	8 × 6

We also note, in the far right column in the table, that the outer two numbers in the matrix sizes determine the dimensions of the new matrix. That is, if an 8-row by 3-column matrix is multiplied by a 3-row by 6-column matrix, the resultant product will be an 8-row by 6-column matrix.

Matrices of the dimensions in the following table may *not* be multiplied:

MATRIX *A* SIZE	MATRIX *B* SIZE
3 × 4	3 × 3
1 × 2	1 × 2
6 × 9	8 × 9
2 × 2	3 × 3

To perform the multiplication process, we take each row of the first matrix and multiply its elements times the numbers in each column of the second matrix. Hence the number in the first row and first column of the new matrix is derived from the product of the first row of the first matrix times the first column of the second matrix. Similarly, the number in the first row and second column of the new matrix is the product of the first row of the first matrix times the second column of the second matrix. This concept is not nearly as confusing as it may sound.

Multiplying numbers

Let us begin by computing the value of matrix C, which is the product of matrix A times matrix B:

$$\text{matrix } A = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} \quad \text{matrix } B = \begin{pmatrix} 4 & 6 \end{pmatrix}$$

This is a legitimate task since matrix A is 3×1 and matrix B is 1×2. The product, matrix C, will have 3 rows and 2 columns (3×2).

Symbolically, the operation is matrix $A \times$ matrix $B =$ matrix C or $AB = C$:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} d & e \end{pmatrix} = \begin{pmatrix} ad & ae \\ bd & be \\ cd & ce \end{pmatrix} \tag{M5-1}$$

Using the actual numbers, we have

$$\begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 4 & 6 \end{pmatrix} = \begin{pmatrix} 20 & 30 \\ 8 & 12 \\ 12 & 18 \end{pmatrix} = \text{matrix } C$$

As a second example, let matrix R be $(6 \;\; 2 \;\; 5)$ and matrix S be

$$\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

Then the product, matrix $T =$ matrix $R \times$ matrix S, will be of dimension 1×1 because we are multiplying a 1×3 matrix by a 3×1 matrix:

$$\begin{matrix} \text{matrix } R \times & \text{matrix } S = & \text{matrix } T \\ (1 \times 3) & (3 \times 1) & (1 \times 1) \end{matrix}$$

$$\begin{pmatrix} a & b & c \end{pmatrix} \times \begin{pmatrix} d \\ e \\ f \end{pmatrix} = (ad + be + cf)$$

$$\begin{pmatrix} 6 & 2 & 5 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = [(6)(3) + (2)(1) + (5)(2)] = (30)$$

To multiply larger matrices, we combine the approaches of the preceding examples:

$$\text{matrix } U = \begin{pmatrix} 6 & 2 \\ 7 & 1 \end{pmatrix} \qquad \text{matrix } V = \begin{pmatrix} 3 & 4 \\ 5 & 8 \end{pmatrix}$$

$$\text{matrix } U \times \text{matrix } V = \text{matrix } Y$$

$$(2 \times 2) \times (2 \times 2) \qquad (2 \times 2)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix} \tag{M5-2}$$

$$\begin{pmatrix} 6 & 2 \\ 7 & 1 \end{pmatrix} \times \begin{pmatrix} 3 & 4 \\ 5 & 8 \end{pmatrix} = \begin{pmatrix} 18 + 10 & 24 + 16 \\ 21 + 5 & 28 + 8 \end{pmatrix} = \begin{pmatrix} 28 & 40 \\ 26 & 36 \end{pmatrix}$$

To introduce a special type of matrix, called the *identity matrix*, let's try a final multiplication example:

$$\text{matrix } H = \begin{pmatrix} 4 & 7 \\ 2 & 3 \end{pmatrix} \qquad \text{matrix } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{matrix } H \times \text{matrix } I = \text{matrix } J$$

$$\begin{pmatrix} 4 & 7 \\ 2 & 3 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 + 0 & 0 + 7 \\ 2 + 0 & 0 + 3 \end{pmatrix} = \begin{pmatrix} 4 & 7 \\ 2 & 3 \end{pmatrix}$$

The identity matrix

Matrix I is called an *identity matrix*. An identity matrix has 1s on its diagonal and 0s in all other positions. When multiplied by any matrix of the same square dimensions, it yields the original matrix. So in this case, matrix J = matrix H.

Matrix multiplication can also be useful in performing business computations.

An example

Blank Plumbing and Heating is about to bid on three contract jobs: to install plumbing fixtures in a new university dormitory, an office building, and an apartment complex. The number of toilets, sinks, and bathtubs needed at each project is summarized in matrix notation as follows. The cost per plumbing fixture is also given. Matrix multiplication can be used to provide an estimate of total cost of fixtures at each job.

PROJECT	DEMAND		
	TOILETS	SINKS	BATHTUBS
Dormitory	5	10	2
Office	20	20	0
Apartments	15	30	15

COST/UNIT	
Toilet	$40
Sink	$25
Bathtub	$50

Job demand matrix × fixture cost matrix = job cost matrix

$$\underset{(3 \times 3)}{\begin{pmatrix} 5 & 10 & 2 \\ 20 & 20 & 0 \\ 15 & 30 & 15 \end{pmatrix}} \times \underset{(3 \times 1)}{\begin{pmatrix} \$40 \\ \$25 \\ \$50 \end{pmatrix}} = \underset{(3 \times 1)}{\begin{pmatrix} \$200 + 250 + 100 \\ \$800 + 500 + 0 \\ \$600 + 750 + 750 \end{pmatrix}} = \begin{pmatrix} \$550 \\ \$1{,}300 \\ \$2{,}100 \end{pmatrix}$$

Hence Blank Plumbing can expect to spend $550 on fixtures at the dormitory project, $1,300 at the office building, and $2,100 at the apartment complex.

Matrix Notation for Systems of Equations

The use of matrices is helpful in representing a system of equations. For example, the system

$$2X_1 + 3X_2 = 24$$
$$4X_1 + 2X_2 = 36$$

can be written as

$$\begin{pmatrix} 2 & 3 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 24 \\ 36 \end{pmatrix}$$

In general, we can express a system of equations as

$$AX = B$$

Matrix Transpose

The *transpose* of a matrix is a means of presenting data in different form. To create the transpose of a given matrix, we simply interchange the rows with the columns. Hence, the first row of a matrix becomes its first column, the second row becomes the second column, and so on.

Two matrices are transposed here:

$$\text{matrix } A = \begin{pmatrix} 5 & 2 & 6 \\ 3 & 0 & 9 \\ 1 & 4 & 8 \end{pmatrix}$$

$$\text{transpose of matrix } A = \begin{pmatrix} 5 & 3 & 1 \\ 2 & 0 & 4 \\ 6 & 9 & 8 \end{pmatrix}$$

$$\text{matrix } B = \begin{pmatrix} 2 & 7 & 0 & 3 \\ 8 & 5 & 6 & 4 \end{pmatrix}$$

$$\text{transpose of matrix } B = \begin{pmatrix} 2 & 8 \\ 7 & 5 \\ 0 & 6 \\ 3 & 4 \end{pmatrix}$$

M5.3 DETERMINANTS, COFACTORS, AND ADJOINTS

There are other important concepts related to matrices. These include the determinant, cofactor and adjoint of a matrix.

Determinants

A *determinant* is a value associated with a square matrix. As a mathematical tool, determinants are of value in helping to solve a series of simultaneous equations.

A 2-row by 2-column (2 × 2) determinant can be expressed by enclosing vertical lines around the matrix, as shown here:

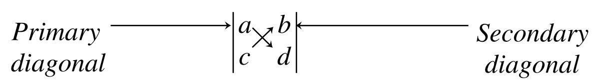
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Similarly, a 3 × 3 determinant is indicated as

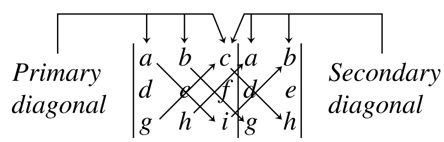
$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

One common procedure for finding the determinant of a 2 × 2 or 3 × 3 matrix is to draw its primary and secondary diagonals. In the case of a 2 × 2 determinant, the value is found by multiplying the numbers on the primary diagonal and subtracting from that product the product of the numbers on the secondary diagonal:

$$\text{value} = (a)(d) - (c)(b)$$



For a 3 × 3 matrix, we redraw the first two columns to help visualize all diagonals and follow a similar procedure.



$$\text{Value} = \begin{bmatrix} \text{1st primary diagonal product } (aei) + \\ \text{2nd primary diagonal product } (bfg) + \\ \text{3rd primary diagonal product } (cdh) \end{bmatrix}$$

$$- \begin{bmatrix} \text{1st secondary diagonal product } (gec) + \\ \text{2nd secondary diagonal product } (hfa) + \\ \text{3rd secondary diagonal product } (idb) \end{bmatrix}$$

$$= aei + bfg + cdh - gec - hfa - idb$$

Let's use this approach to find the numerical values of the following 2 × 2 and 3 × 3 determinants:

$$\text{(a)}\ \begin{vmatrix} 2 & 5 \\ 1 & 8 \end{vmatrix} \qquad \text{(b)}\ \begin{vmatrix} 3 & 1 & 2 \\ 2 & 5 & 1 \\ 4 & -2 & -1 \end{vmatrix}$$

$$\text{(a)}\ \begin{vmatrix} 2 & 5 \\ 1 & 8 \end{vmatrix} \qquad \text{Value} = (2)(8) - (1)(5) = 11$$

$$\text{(b)}\ \left|\begin{matrix} 3 & 1 & 2 \\ 2 & 5 & 1 \\ 4 & -2 & -1 \end{matrix}\right|\begin{matrix} 3 & 1 \\ 2 & 5 \\ 4 & -2 \end{matrix}\right|$$

$$\text{Value} = (3)(5)(-1) + (1)(1)(4) + (2)(2)(-2) - (4)(5)(2) - (-2)(1)(3) - (-1)(2)(1)$$

$$= -15 + 4 - 8 - 40 + 6 + 2 = -51$$

A set of *simultaneous equations* can be solved through the use of determinants by setting up a ratio of two special determinants for each unknown variable. This fairly easy procedure is best illustrated with an example.

Given the three simultaneous equations

$$2X + 3Y + 1Z = 10$$

$$4X - 1Y - 2Z = 8$$

$$5X + 2Y - 3Z = 6$$

we can structure determinants to help solve for unknown quantities *X, Y,* and *Z.*

$$X = \frac{\begin{vmatrix} 10 & 3 & 1 \\ 8 & -1 & -2 \\ 6 & 2 & -3 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 1 \\ 4 & -1 & -2 \\ 5 & 2 & -3 \end{vmatrix}}$$

Coefficients for right-hand side
Coefficients for Y
Coefficients for Z
Numerator determinant, in which column with Xs is replaced by column of numbers to the right-hand-side of the equal sign.
Denominator determinant, in which coefficients of all unknown variables are listed (all columns to the left of the equal sign)
Coefficients for Z
Coefficients for Y
Coefficients for X

$$Y = \frac{\begin{vmatrix} 2 & 10 & 1 \\ 4 & 8 & -2 \\ 5 & 6 & -3 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 1 \\ 4 & -1 & -2 \\ 5 & 2 & -3 \end{vmatrix}}$$

← *Numerator determinant, in which column with* Y*s is replaced by right-hand-side numbers*
← *Denominator determinant stays the same regardless of which variable we are solving for*

$$Z = \frac{\begin{vmatrix} 2 & 3 & 10 \\ 4 & -1 & 8 \\ 5 & 2 & 6 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 1 \\ 4 & -1 & -2 \\ 5 & 2 & -3 \end{vmatrix}}$$

← *Numerator determinant, in which column with* Z*s is replaced by right-hand-side numbers*
← *Denominator determinant, again the as same when solving for* X *and* Y

Determining the values of *X, Y,* and *Z* now involves finding the numerical values of the four separate determinants using the method shown earlier in this supplement.

$$X = \frac{\text{numerical value of numerator determinant}}{\text{numerical value of denominator determinant}}$$

$$= \frac{128}{33} = 3.88$$

$$Y = \frac{-20}{33} = -0.61$$

$$Z = \frac{134}{33} = 4.06$$

To verify that $X = 3.88$, $Y = -0.61$, and $Z = 4.06$, we may choose any one of the original three simultaneous equations and insert these numbers. For example,

$$2X + 3Y + 1Z = 10$$

$$2(3.88) + 3(-0.61) + 1(4.06) = 7.76 - 1.83 + 4.06$$

$$= 10$$

Matrix of Cofactors and Adjoint

Two more useful concepts in the mathematics of matrices are the *matrix of cofactors* and the *adjoint of a matrix.* A *cofactor* is defined as the set of numbers that remains after a given row and column have been taken out of a matrix. An *adjoint* is simply the transpose of the matrix of cofactors. The real value of the two concepts lies in the usefulness in forming the inverse of a matrix—something that we investigate in the next section.

To compute the matrix of cofactors for a particular matrix, we follow six steps:

Six Steps in Computing a Matrix of Cofactors

1. Select an element in the original matrix.
2. Draw a line through the row and column of the element selected. The numbers uncovered represent the cofactor for that element.
3. Calculate the value of the determinant of the cofactor.
4. Add together the location numbers of the row and column crossed out in step 2. If the sum is even, the sign of the determinant's value (from step 3) does not change. If the sum is an odd number, change the sign of the determinant's value.
5. The number just computed becomes an entry in the matrix of cofactors; it is located in the same position as the element selected in step 1.
6. Return to step 1 and continue until all elements in the original matrix have been replaced by their cofactor values.

TABLE M5.1

Matrix of Cofactor Calculations

ELEMENT REMOVED	COFACTORS	DETERMINANT OF COFACTORS	VALUE OF COFACTOR
Row 1, column 1	$\begin{pmatrix} 0 & 3 \\ 1 & 8 \end{pmatrix}$	$\begin{vmatrix} 0 & 3 \\ 1 & 8 \end{vmatrix} = -3$	−3 (sign not changed)
Row 1, column 2	$\begin{pmatrix} 2 & 3 \\ 4 & 8 \end{pmatrix}$	$\begin{vmatrix} 2 & 3 \\ 4 & 8 \end{vmatrix} = 4$	−4 (sign changed)
Row 1, column 3	$\begin{pmatrix} 2 & 0 \\ 4 & 1 \end{pmatrix}$	$\begin{vmatrix} 2 & 0 \\ 4 & 1 \end{vmatrix} = 2$	2 (sign not changed)
Row 2, column 1	$\begin{pmatrix} 7 & 5 \\ 1 & 8 \end{pmatrix}$	$\begin{vmatrix} 7 & 5 \\ 1 & 8 \end{vmatrix} = 51$	−51 (sign changed)
Row 2, column 2	$\begin{pmatrix} 3 & 5 \\ 4 & 8 \end{pmatrix}$	$\begin{vmatrix} 3 & 5 \\ 4 & 8 \end{vmatrix} = 4$	4 (sign not changed)
Row 2, column 3	$\begin{pmatrix} 3 & 7 \\ 4 & 1 \end{pmatrix}$	$\begin{vmatrix} 3 & 7 \\ 4 & 1 \end{vmatrix} = -25$	25 (sign changed)
Row 3, column 1	$\begin{pmatrix} 7 & 5 \\ 0 & 3 \end{pmatrix}$	$\begin{vmatrix} 7 & 5 \\ 0 & 3 \end{vmatrix} = 21$	21 (sign not changed)
Row 3, column 2	$\begin{pmatrix} 3 & 5 \\ 2 & 3 \end{pmatrix}$	$\begin{vmatrix} 3 & 5 \\ 2 & 3 \end{vmatrix} = -1$	1 (sign changed)
Row 3, column 3	$\begin{pmatrix} 3 & 7 \\ 2 & 0 \end{pmatrix}$	$\begin{vmatrix} 3 & 7 \\ 2 & 0 \end{vmatrix} = -14$	−14 (sign not changed)

Let's compute the matrix of cofactors, and then the adjoint, for the following matrix, using Table M5.1 to help in the calculations:

$$\text{original matrix} = \begin{pmatrix} 3 & 7 & 5 \\ 2 & 0 & 3 \\ 4 & 1 & 8 \end{pmatrix}$$

$$\text{matrix of cofactors} = \begin{pmatrix} -3 & -4 & 2 \\ -51 & 4 & 25 \\ 21 & 1 & -14 \end{pmatrix}$$

$$\text{adjoint of the matrix} = \begin{pmatrix} -3 & -51 & 21 \\ -4 & 4 & 1 \\ 2 & 25 & -14 \end{pmatrix}$$

(from Table M5.1)

M5.4 FINDING THE INVERSE OF A MATRIX

The *inverse* of a matrix is a unique matrix of the same dimensions which, when multiplied by the original matrix, produces a *unit* or *identity* matrix. For example, if A is any 2×2 matrix and its inverse is denoted A^{-1}, then

$$A \times A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \text{identity matrix} \quad \text{(M5-3)}$$

The adjoint of a matrix is extremely helpful in forming the inverse of the original matrix. We simply compute the value of the determinant of the original matrix and divide each term of the adjoint by this value.

To find the inverse of the matrix just presented, we need to know the adjoint (already computed) and the value of the determinant of the original matrix:

$$\begin{pmatrix} 3 & 7 & 5 \\ 2 & 0 & 3 \\ 4 & 1 & 8 \end{pmatrix} = \text{original matrix}$$

Determining the value of the determinant

Value of determinant:

$$\left| \begin{array}{ccc|cc} 3 & 7 & 5 & 3 & 7 \\ 2 & 0 & 3 & 2 & 0 \\ 4 & 1 & 8 & 4 & 1 \end{array} \right.$$

$$\text{value} = 0 + 84 + 10 - 0 - 9 - 112 = -27$$

The inverse is found by dividing each element in the adjoint by −27:

$$\text{inverse} = \begin{pmatrix} {}^{-3}\!/_{-27} & {}^{-51}\!/_{-27} & {}^{21}\!/_{-27} \\ {}^{-4}\!/_{-27} & {}^{4}\!/_{-27} & {}^{1}\!/_{-27} \\ {}^{2}\!/_{-27} & {}^{25}\!/_{-27} & {}^{-14}\!/_{-27} \end{pmatrix}$$

$$= \begin{pmatrix} {}^{3}\!/_{27} & {}^{51}\!/_{27} & {}^{-21}\!/_{27} \\ {}^{4}\!/_{27} & {}^{-4}\!/_{27} & {}^{-1}\!/_{27} \\ {}^{-2}\!/_{27} & {}^{-25}\!/_{27} & {}^{14}\!/_{27} \end{pmatrix}$$

We can verify that this is indeed the correct inverse of the original matrix by multiplying the original matrix times the inverse:

$$\begin{matrix} \text{original matrix} & \times & \text{inverse} & = & \text{identity matrix} \end{matrix}$$

$$\begin{pmatrix} 3 & 7 & 5 \\ 2 & 0 & 3 \\ 4 & 1 & 8 \end{pmatrix} \times \begin{pmatrix} {}^{3}\!/_{27} & {}^{51}\!/_{27} & {}^{-21}\!/_{27} \\ {}^{4}\!/_{27} & {}^{-4}\!/_{27} & {}^{-1}\!/_{27} \\ {}^{-2}\!/_{27} & {}^{-25}\!/_{27} & {}^{14}\!/_{27} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If this process is applied to a 2×2 matrix, the inverse is easily found, as shown with the following matrix.

$$\text{original matrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\text{determinant value of original matrix} = ad - cb$$

$$\text{matrix of cofactors} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

$$\text{adjoint of the matrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Since the inverse of the matrix is equal to the adjoint divided by the determinant, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} \dfrac{d}{ad - cb} & \dfrac{-b}{ad - cb} \\ \dfrac{-c}{ad - cb} & \dfrac{a}{ad - cb} \end{pmatrix} \qquad \text{(M5-4)}$$

For example, if

$$\text{original matrix} = \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}$$

$$\text{determinant value} = 1(8) - 3(2) = 2$$

then

$$\text{inverse} = \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}^{-1} = \begin{pmatrix} 8/2 & -2/2 \\ -3/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ -1.5 & 0.5 \end{pmatrix}$$

SUMMARY

This module contains a brief presentation of matrices and common operations associated with matrices. The discussion includes matrix addition, subtraction, and multiplication. We demonstrate how to represent a system of equations with matrix notation. This is used with the linear programming simplex algorithm in Chapter 9. We show how determinants are used in finding the inverse of a matrix and solving a series of simultaneous equations. Interchanging the rows and columns of a matrix results in the transpose of that matrix. Cofactors and adjoints are used to develop the inverse of a matrix.

GLOSSARY

Adjoint. The transpose of a matrix of cofactors.

Determinant. A unique numerical value associated with a square matrix.

Identity Matrix. A square matrix with 1s on its diagonal and 0s in all other positions.

Inverse. A unique matrix that can be multiplied by the original matrix to create an identity matrix.

Matrix. An array of numbers that can be used to present or summarize business data.

Matrix of Cofactors. The determinants of the numbers remaining in a matrix after a given row and column have been removed.

Simultaneous Equations. A series of equations that must be solved at the same time.

Transpose. The interchange of rows and columns in a matrix.

KEY EQUATIONS

(M5-1) $$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} d & e \end{pmatrix} = \begin{pmatrix} ad & ae \\ bd & be \\ cd & ce \end{pmatrix}$$

Multiplying matrices.

(M5-2) $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$

Multiplying 2×2 matrices.

(M5-3) $$A \times A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The identity matrix.

(M5-4) $$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} \dfrac{d}{ad - cb} & \dfrac{-b}{ad - cb} \\ \dfrac{-c}{ad - cb} & \dfrac{a}{ad - cb} \end{pmatrix}$$

The inverse of a 2×2 matrix.

SELF-TEST

- **Before taking the self-test, refer back to the learning objectives at the beginning of the module and the glossary at the end of the module.**
- **Use the key at the back of the book to correct your answers.**
- **Restudy pages that correspond to any questions that you answered incorrectly or material you feel uncertain about.**

1. The value of the determinant $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$ is
   a. 2.
   b. 10.
   c. − 2.
   d. −5.
   e. 14.
2. To add two matrices,
   a. the first matrix must have the same number of rows and columns as the second matrix.
   b. the number of rows in the second matrix must equal the number of the columns in the first matrix.
   c. the number of columns in the second matrix must equal the number of the rows in the first matrix.
   d. both matrices must be square matrices.
3. In a 2 × 2 matrix, the determinant value is
   a. equal to the sum of all the values in the matrix.
   b. found by multiplying the numbers on the primary diagonal and subtracting from that the product of the numbers on the secondary diagonal.
   c. found by turning each row of the matrix into a column.
   d. always equal to 4.
4. If matrix *A* is a 2 × 3 matrix, it can be multiplied by matrix *B* to obtain matrix *AB* only if matrix *B* has
   a. 2 rows.
   b. 2 columns
   c. 3 rows.
   d. 3 columns.
5. If matrix *A* is a 3 × 4 matrix, the transpose of matrix *A* will be a
   a. 3 × 4 matrix.
   b. 4 × 3 matrix.
   c. 3 × 3 matrix.
   d. 4 × 4 matrix.
6. When a matrix is multiplied by an identity matrix, the result will be
   a. the original matrix.
   b. an identity matrix.
   c. the inverse of the original matrix.
   d. the transpose of the original matrix.
7. An identity matrix
   a. has 1s on its diagonal.
   b. has 0s in all positions not on a diagonal.
   c. can be multiplied by any matrix of the same dimensions.
   d. is square in size.
   e. all of the above.
8. When the inverse of a matrix is multiplied by the original matrix, it produces
   a. the matrix of cofactors.
   b. the adjoint of the matrix.
   c. the transpose.
   d. the identity matrix.
   e. none of the above.

DISCUSSION QUESTIONS AND PROBLEMS

Discussion Questions

M5-1 List some of the ways that matrices are used.

M5-2 Explain how to calculate the determinant of a matrix.

M5-3 What are some of the uses of determinants?

M5-4 Explain how to develop a matrix of cofactors.

M5-5 What is the adjoint of a matrix?

M5-6 How is the adjoint used in finding the inverse of a matrix?

Problems

• **M5-7** Find the numerical values of the following determinants.

(a) $\begin{vmatrix} 6 & 3 \\ -5 & 2 \end{vmatrix}$

(b) $\begin{vmatrix} 3 & 7 & -6 \\ 1 & -1 & 2 \\ 4 & 3 & -2 \end{vmatrix}$

: **M5-8** Use determinants to solve the following set of simultaneous equations.

$$5X + 2Y + 3Z = 4$$

$$2X + 3Y + 1Z = 2$$

$$3X + 1Y + 2Z = 3$$

• **M5-9** Use matrices to write the system of equations in Problem M5-8.

• **M5-10** Perform the following operations.

(a) Add matrix *A* to matrix *B*.
(b) Subtract matrix *A* from matrix *B*.
(c) Add matrix *C* to matrix *D*.
(d) Add matrix *C* to matrix *A*.

$$\text{matrix } A = \begin{pmatrix} 2 & 4 & 1 \\ 3 & 8 & 7 \end{pmatrix} \qquad \text{matrix } C = \begin{pmatrix} 3 & 6 & 9 \\ 7 & 8 & 1 \\ 9 & 2 & 4 \end{pmatrix}$$

$$\text{matrix } B = \begin{pmatrix} 7 & 6 & 5 \\ 0 & 1 & 2 \end{pmatrix} \qquad \text{matrix } D = \begin{pmatrix} 5 & 1 & 6 \\ 4 & 0 & 6 \\ 3 & 1 & 5 \end{pmatrix}$$

M5-11 Using the matrices in Problem M5-10, determine which of the following operations are possible. If the operation is possible, give the size of the resulting matrix.

(a) $B \times C$
(b) $C \times B$
(c) $A \times B$
(d) $C \times D$
(e) $D \times C$

M5-12 Perform the following matrix multiplications.

(a) Matrix C = matrix $A \times$ matrix B
(b) Matrix G = matrix $E \times$ matrix F
(c) Matrix T = matrix $R \times$ matrix S
(d) Matrix Z = matrix $W \times$ matrix Y

$$\text{matrix } A = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \text{matrix } B = \begin{pmatrix} 3 & 4 & 5 \end{pmatrix}$$

$$\text{matrix } E = \begin{pmatrix} 5 & 2 & 6 & 1 \end{pmatrix} \qquad \text{matrix } F = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 0 \end{pmatrix}$$

$$\text{matrix } R = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \qquad \text{matrix } S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{matrix } W = \begin{pmatrix} 3 & 5 \\ 2 & 1 \\ 4 & 4 \end{pmatrix} \qquad \text{matrix } Y = \begin{pmatrix} 1 & 4 & 5 & 1 \\ 2 & 3 & 6 & 5 \end{pmatrix}$$

M5-13 RLB Electrical Contracting, Inc., bids on the same three jobs as Blank Plumbing (Section M5.3). RLB must supply wiring, conduits, electrical wall fixtures, and lighting fixtures. The following are needed supplies and their costs per unit:

	Demand			
Project	Wiring (Rolls)	Conduits	Wall Fixtures	Lighting Fixtures
Dormitory	50	100	10	20
Office	70	80	20	30
Apartments	20	50	30	10

Item	Cost/Unit ($)
Wiring	1.00
Conduits	2.00
Wall fixtures	3.00
Lighting fixtures	5.00

Use matrix multiplication to compute the cost of materials at each job site.

M5-14 Transpose matrices R and S.

$$\text{matrix } R = \begin{pmatrix} 6 & 8 & 2 & 2 \\ 1 & 0 & 5 & 7 \\ 6 & 4 & 3 & 1 \\ 3 & 1 & 2 & 7 \end{pmatrix}$$

$$\text{matrix } S = \begin{pmatrix} 3 & 1 \\ 2 & 2 \\ 5 & 4 \end{pmatrix}$$

M5-15 Find the matrix of cofactors and adjoint of the following matrix:

$$\begin{pmatrix} 1 & 4 & 7 \\ 2 & 0 & 8 \\ 3 & 6 & 9 \end{pmatrix}$$

M5-16 Find the inverse of original matrix of Problem M5-15 and verify its correctness.

M5-17 Write the following as a system of equations:

$$\begin{pmatrix} 5 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 240 \\ 320 \end{pmatrix}$$

M5-18 Write the following as a system of equations:

$$\begin{pmatrix} 0 & 4 & 3 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 28 \\ 16 \end{pmatrix}$$

BIBLIOGRAPHY

Barnett, Raymond, Michael Ziegler, and Karl Byleen. *College Mathematics for Business, Economics, Life Sciences, and Social Sciences, 9/e*. Upper Saddle River, NJ: Prentice Hall, 2002.

Haeussler, Ernest and Richard Paul. *Introductory Mathematical Analysis for Business, Economics and the Life and Social Sciences, 10/e*. Upper Saddle River, NJ: Prentice Hall, 2002.

APPENDIX M5.1: USING EXCEL FOR MATRIX CALCULATIONS

Three Excel functions are helpful with matrices: MMULT for matrix multiplication, MINVERSE for finding the inverse of the matrix, and MDETERM for finding the determinant of a matrix.

To multiply matrices in Excel, use MMULT as follows:

1. Highlight all the cells that will contain the resulting matrix.
2. Type = **MMULT(matrix1, matrix2)**, where matrix 1 and matrix 2 are the cell ranges for the two matrices being multiplied.
3. Instead of just pressing ENTER, press CTRL+SHIFT+ENTER simultaneously.

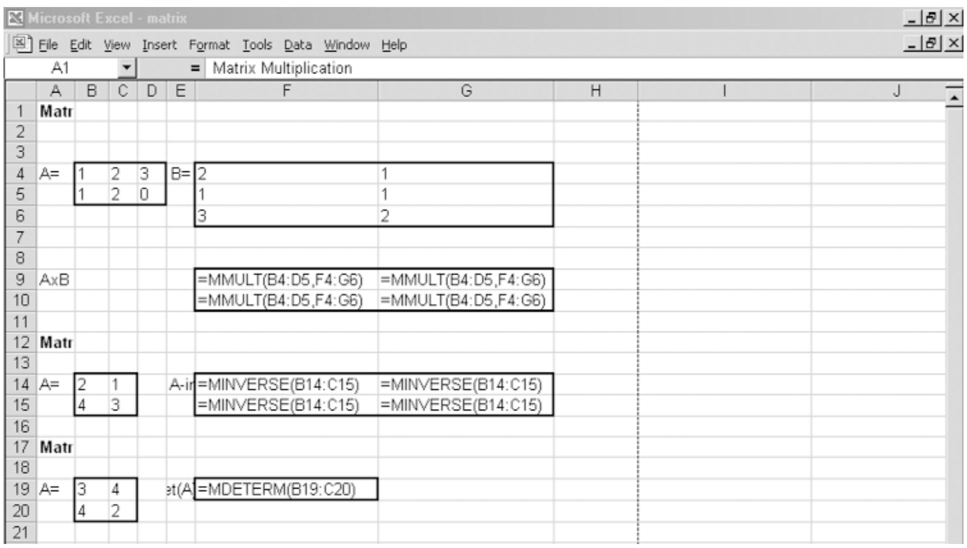
Pressing CTRL+SHIFT+ENTER indicates that a matrix operation is being performed so that all cells in the matrix are changed accordingly. Program M5.1A provides an example of this in cells F9:G10. The output is shown in Program M5.1B.

The MINVERSE function is illustrated in cells F14:G15 of Program M5.1A. Remember that the entire range of the matrix (F14:G15) must be highlighted before entering the MINVERSE function. Pressing SHIFT+CTRL+ENTER provides the final result shown in Program M5.1B.

The MDETERM function is used to find the determinant of a matrix. This is illustrated in cell F19 of Program M5.1A. The numerical result is shown in Program M5.1B.

PROGRAM M5.1A

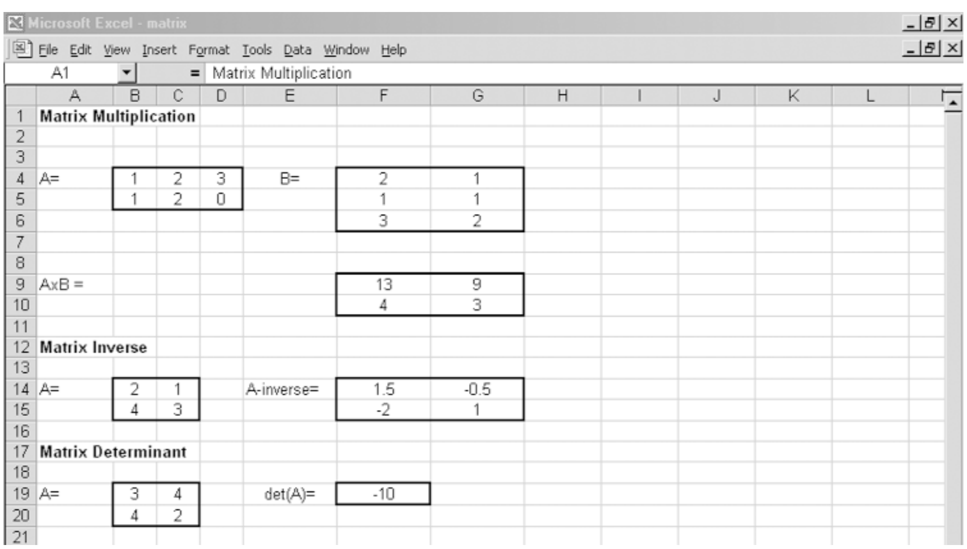
Excel Formulas for Matrix Operations


Microsoft Excel - matrix
File Edit View Insert Format Tools Data Window Help
A1 = Matrix Multiplication

	A	B	C	D	E	F	G
1	Matr						
2							
3							
4	A=	1	2	3	B=	2	1
5		1	2	0		1	1
6						3	2
7							
8							
9	AxB					=MMULT(B4:D5,F4:G6)	=MMULT(B4:D5,F4:G6)
10						=MMULT(B4:D5,F4:G6)	=MMULT(B4:D5,F4:G6)
11							
12	Matr						
13							
14	A=	2	1		A-ir	=MINVERSE(B14:C15)	=MINVERSE(B14:C15)
15		4	3			=MINVERSE(B14:C15)	=MINVERSE(B14:C15)
16							
17	Matr						
18							
19	A=	3	4		et(A	=MDETERM(B19:C20)	
20		4	2				
21							

PROGRAM M5.1B

Excel Output for Matrix Operations


Microsoft Excel - matrix
File Edit View Insert Format Tools Data Window Help
A1 = Matrix Multiplication

	A	B	C	D	E	F	G
1	Matrix Multiplication						
2							
3							
4	A=	1	2	3	B=	2	1
5		1	2	0		1	1
6						3	2
7							
8							
9	AxB =					13	9
10						4	3
11							
12	Matrix Inverse						
13							
14	A=	2	1		A-inverse=	1.5	-0.5
15		4	3			-2	1
16							
17	Matrix Determinant						
18							
19	A=	3	4		det(A)=	-10	
20		4	2				
21							